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THE URYSOHN UNIVERSAL METRIC SPACE IS HOMEOMORPHIC TO A HILBERT SPACE

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Abstract. The Urysohn universal metric space U is characterized up to isometry by the following properties: (1) U is complete and separable; (2) U contains an isometric copy of every separable metric space; (3) every isometry between two finite subsets of U can be extended to an isometry of U onto itself. We show that U is homeomorphic to the Hilbert space l_2 (or to the countable power of the real line).

1. Introduction

The Urysohn universal metric space U is characterized up to isometry by the following properties: (1) U is complete and separable; (2) U contains an isometric copy of every separable metric space; (3) every isometry between two finite subsets of U can be extended to an isometry of U onto itself. (An *isometry* is a distance-preserving bijection; an *isometric embedding* is a distance-preserving injection.) The aim of the present paper is to show that the Urysohn space U is homeomorphic to a Hilbert space (equivalently, to the countable power of the real line). This answers a question raised by Bogatyĭ, Pestov and Vershik.

There is another characterization of U. Let us say that a metric space M is *injective with respect to finite spaces*, or *finitely injective* for short, if for every finite metric space L, every subspace $K \subset L$ and every isometric embedding $f : K \to M$ there exists an isometric embedding $\bar{f} : L \to M$ which extends f. Define *compactly injective* metric spaces similarly, considering compact (rather than finite) spaces K and L. If a metric space M contains an isometric copy of every finite metric space and satisfies the condition (3) above, then M is finitely injective. Indeed, given finite metric spaces $K \subset L$ and an isometric embedding $f : K \to M$, we find an isometric embedding $g : L \to M$ and extend the isometry $gf^{-1} : f(K) \to g(K)$ to an isometry h of M onto itself. Then $h^{-1}g : L \to M$ is an isometric embedding of L which extends f. Conversely, let M be a finitely injective metric space. Then every countable metric space admits an isometric embedding into M (use induction). If M is also complete, it follows that M contains an isometric copy of every separable metric space. Assume now that M is also separable, and let $f : K \to L$ be an isometry between two finite subsets of M. Enumerating points of a dense countable subset of M and using the back-and-forth method we can extend f to an isometry between two dense countable subsets of M and then to an isometry of M onto itself. The same argument shows

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that any two complete separable finitely injective metric spaces are isometric. Thus the Urysohn space U is the unique (up to isometry) metric space which is complete, separable and finitely injective.

The existence of U was proved by Urysohn [8, 9]. An easier construction was found some 50 years later by Katětov [2], who also gave an example of a non-complete separable metric space satisfying the conditions (2) and (3) above, thus answering a question of Urysohn. Katětov's construction was used in [10, 11, 12] to prove that the topological group $\mathrm{Is}\,(U)$ of all isometries of U is universal, in the sense that it contains an isomorphic copy of every topological group with a countable base. A deep result concerning the group $G = \mathrm{Is}\,(U)$ was established by V.Pestov: the group G is extremely amenable, i.e., every compact space with a continuous action of G has a G-fixed point [6, 3].

A.M.Vershik showed that the space U can be obtained as the completion of a countable metric space equipped with a metric which is either "random" or generic in the sense of Baire category [13, 14, 15].

Bogatyĭ [1] proved that any isometry between two compact subsets of U can be extended to an isomerty of U onto itself. It follows by the same argument that we used above for finitely injective spaces that U is compactly injective (and is the unique complete separable compactly injective metric space). Using this, we deduce our Main Theorem from Toruńczyk's Criterion [7, 5]: a complete separable metric space M is homeomorphic to the Hilbert space l_2 if and only if M is AR (= absolute retract) and has the discrete approximation property (this notion is defined below). Recall that all infinite-dimensional separable Banach spaces are homeomorphic to each other and to the countable power of the real line.

Given an open cover $\mathcal{U}$ of a space X, two points $x, y \in X$ are said to be *$\mathcal{U}$-close* if there exists $U \in \mathcal{U}$ such that $x, y \in U$. A family of subsets of a space X is *discrete* if every point in X has a neighbourhood which meets at most one member of the family. A metric space M has the *discrete approximation property* if for every sequence $K_1, K_2, \dots$ of compact subspaces of M and every open cover $\mathcal{U}$ of M there exists a sequence of maps $f_i : K_i \to M$ such that for every i and every $x \in K_i$ the points x and $f_i(x)$ are $\mathcal{U}$-close and the sequence $(f_i(K_i))$ is discrete. Equivalently [7], a metric space (M, d) has the discrete approximation property if and only if for every sequence $K_1, K_2, \dots$ of compact subspaces of M and every continuous function h on M with values > 0 there exists a sequence of maps $f_i : K_i \to M$ such that $d(x, f_i(x)) \le h(x)$ for every i and every $x \in K_i$, and the sequence $(f_i(K_i))$ is discrete.

Let us reformulate Toruńczyk's Criterion in the form that is convenient for our purposes. We say that a topological space X is *homotopically trivial* if X has trivial homotopy groups, that is, every map of the n-sphere $S^n = \partial B^{n+1}$ to X admits an extension over the $(n+1)$-ball B^{n+1} $(n = 0, 1, \dots)$. (The term *weakly homotopically trivial* might be more appropriate.) Every contractible space is homotopically trivial; the converse in general is not true. The empty space is homotopically trivial. If a metric space M has a base $\mathcal{B}$ such that for every non-empty finite subfamily $\mathcal{F} \subset \mathcal{B}$ the intersection $\cap\mathcal{F}$ is homotopically trivial, then M is ANR [4, Theorem 5.2.12]. A metric space is AR if and only if it is homotopically trivial and ANR [4, Theorem 5.2.15]. Thus Toruńczyk's Criterion can be reformulated as follows:

Theorem 1.1 (**Toruńczyk's Criterion**). *A complete separable metric space M is homeomorphic to a Hilbert space if and only if the following conditions hold:*

(i) *there is a base $\mathcal{B}$ for M such that $U, V \in \mathcal{B}$ implies $U \cap V \in \mathcal{B}$, and every $U \in \mathcal{B}$ is homotopically trivial;*
(ii) *M is homotopically trivial;*
(iii) *M has the discrete approximation property.*

In the next section we show that the space U satisfies the conditions of this criterion.

2. Proof of the main theorem

Theorem 2.1 (Main Theorem). *The Urysohn universal space U is homeomorphic to a Hilbert space.*

Proof. We check the three conditions of Toruńczyk's criterion.

(a) Let $\mathcal{B}$ be the base for U consisting of all open balls $O(a,r) = \{x \in U : d(x,a) < r\}$ and their finite intersections. We claim that every member $V = \cap_{i=1}^{k} O(a_i, r_i)$ of $\mathcal{B}$ is homotopically trivial. Let a map $f : S^n \to V$ be given. We must construct an extension $\bar{f} : B^{n+1} \to V$.

Every metric space admits an isometric embedding into a normed linear space. Thus we may consider U as a subspace of a Banach space B. Let $V' = \cap_{i=1}^{k} O'(a_i, r_i)$, where $O'(a,r)$ is the open ball centered at a of radius r in the space B. Then $V = V' \cap U$. Being a convex subset of a normed linear space, the space V' is contractible (in fact it is AR [4, Theorem 1.4.13]), so the map $f : S^n \to V$ can be extended to a map $g : B^{n+1} \to V'$. Let $A = \{a_1, \dots, a_k\}$, $K = f(S^n) \cup A$ and $L = g(B^{n+1}) \cup A$. Then K and L are compact, $K \subset L \cap U$. Since U is compactly injective, the identity map of K can be extended to an isometric embedding $h : L \to U$. Let $\bar{f} = hg : B^{n+1} \to U$. Then $\bar{f}$ extends f. The range of $\bar{f}$ is contained in V, since for every $x \in B^{n+1}$ and $i = 1, \dots, k$ we have $d(\bar{f}(x), a_i) = d(h(g(x)), h(a_i)) = d(g(x), a_i) < r_i$ (note that $h(a_i) = a_i$, since $a_i \in K$ and h fixes all points in K).

(b) The space U is homotopically trivial. The proof is the same as above but easier, since we do not have to care about points $a_1, \dots, a_k$.

(c) We prove that U has the discrete approximation property. Let $K_1, \dots, K_n, \dots$ be a sequence of non-empty compact subsets of U, and let h be a continuous function on U with values > 0. We must construct a discrete sequence (L_n) of compact subsets of U and a sequence of maps $f_n : K_n \to L_n$ such that $d(f_n(x), x) \le h(x)$ for every $n \ge 1$ and $x \in K_n$.

We'll need the notion of union of two metric spaces with a subspace amalgamated. Suppose that M_1, M_2, A are metric spaces, $A \neq \emptyset$, and isometric embeddings $f_i : A \to M_i$, $i = 1, 2$, are given. The union M of M_1 and M_2 with the subspace A amalgamated is characterized by the following properties: there exist isometric embeddings $h_i : M_i \to M$ such that $M = h_1(M_1) \cup h_2(M_2)$, $h_1 f_1 = h_2 f_2$, and for every $x \in M_1 \setminus f_1(A)$, $y \in M_2 \setminus f_2(A)$

$$d(h_1(x), h_2(y)) = \inf\{d_1(x, f_1(z)) + d_2(f_2(z), y) : z \in A\},$$

where d, d_1, d_2 are the metrics on M, M_1, M_2, respectively. It is easy to see that such a space M exists and in the obvious sense is unique.

Let $N_i \subset K_i \times \mathbf{R}$ be the union of $K_i \times \{0\}$ and the graph of the restriction of h on K_i. Equip $K_i \times \mathbf{R}$ with the metric ρ defined by

$$\rho((x,t),(y,s)) = d(x,y) + |s-t|,$$

and consider the induced metric on N_i.

We now construct a sequence (L_n) of compact subsets of U by induction. Suppose the sets L_i have been defined for $i < n$. We define L_n. Consider two compact metric spaces: $K_n \cup \bigcup_{i<n} L_i$ and N_n. Since K_n lies in the first space and has a natural embedding into the second one (we mean the embedding $x \mapsto (x,0)$), we can construct their union with the subspace K_n amalgamated. Write this union as $P = \bigcup_{i<n} L_i \cup K_n \cup \Gamma_n$, where $\Gamma_n = \{(x,h(x)) : x \in K_n\}$ is the graph of $h \restriction K_n$. Since U is compactly injective, there exists an isometric embedding $\phi : P \to U$ which is identity on each L_i $(i < n)$ and on K_n. Let $L_n = \phi(\Gamma_n)$. Let $f_n : K_n \to L_n$ be the composition of the map $x \mapsto (x, h(x))$ from K_n onto Γ_n and ϕ. For every $x \in K_n$ the distance from $(x,0)$ to $(x,h(x))$ in N_n is equal to $h(x)$, hence the distance from x to $(x,h(x))$ in P and the distance from x to $f_n(x)$ in U also are equal to $h(x)$. Thus f_n moves every $x \in K_n$ by $h(x)$.

Note that for every $x \in K_n$ and $y \in \bigcup_{i<n} L_i$ the distance from $f_n(x)$ to y is $\geq h(x)$. Indeed, by our construction this distance is equal to the distance from $(x,h(x)) \in \Gamma_n$ to y in P and thus also to

$$\begin{aligned}\inf\{d(y,z) + \rho((z,0),(x,h(x))) : z \in K_n\} &= \inf\{d(y,z) + d(z,x) + h(x) : z \in K_n\} \\ &= d(y,x) + h(x) \geq h(x).\end{aligned}$$

To conclude the proof, we must show that the sequence (L_n) is discrete. Assume the contrary. Since the sequence (L_n) is disjoint, there exists an infinite set A of positive integers and points $y_i \in L_i$ $(i \in A)$ such that the sequence $\{y_i : i \in A\}$ converges to some $p \in U$. Write $y_i = f_i(x_i)$, where $x_i \in K_i$. The distance from y_n to $\{y_i : i < n,\ i \in A\} \subset \bigcup_{i<n} L_i$ tends to zero as $n \in A$ tends to infinity. On the other hand, we saw in the preceding paragraph that this distance is $\geq h(x_n)$. Therefore the sequence $\{h(x_n) : n \in A\}$ tends to zero. Since $d(x_n, y_n) = d(x_n, f_n(x_n)) = h(x_n) \to 0$ and $y_n \to p$, it follows that $x_n \to p$. But this contradicts the continuity of h at p: we have $h(p) > 0$, $x_n \to p$ and $h(x_n) \to 0$. $\square$

References

[1] S.A. Bogatyĭ, *Metrically homogeneous spaces* (Russian), Uspekhi Mat. Nauk **57** (2002), no. 2(344), 3–22; translation in Russian Math. Surveys **57** (2002), no. 2, 221–240.

[2] M. Katětov, *On universal metric spaces,* in book: *General Topology and its Relations to Modern Analysis and Algebra VI: Proceedings of the Sixth Prague Topological Symposium 1986*, Z. Frolík (editor), Berlin: Heldermann Verlag, 1988, pp. 323–330.

[3] A.S. Kechris, V.G. Pestov, and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, arXiv: math.LO/0305241

[4] J. van Mill, *Infinite-dimensional topology: Prerequisites and Introduction,* North-Holland, 1989.

[5] M. Bestvina, P. Bowers, J. Mogilski, J. Walsh, *Characterization of Hilbert space manifolds revisited*, Topol. Appl. **24** (1986), 53–69.

[6] V. Pestov, *Ramsey – Milman phenomenon, Urysohn metric spaces, and extremely amenable groups*, Israel J. Math. **127** (2002), 317–357; arXiv: math.FA/0004010

[7] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. **111** (1981), no. 3, 247–262.

[8] P. Urysohn, *Sur un espace métrique universel,* C. R. Acad. Sci. Paris **180** (1925), 803–806.

[9] P. Urysohn, *Sur un espace métrique universel,* Bull. Sci. Math. **51** (1927), 43–64 and 74–90.

[10] V. Uspenskij, *On the group of isometries of the Urysohn universal metric space,* Comment. Math. Univ. Carolinae **31** (1990), no. 1, 181–182.

[11] V. Uspenskij, *On subgroups of minimal topological groups*, arXiv: math.GN/0004119

[12] V. Uspenskij, *Compactifications of topological groups,* Proceedings of the Ninth Prague Topological Symposium (Prague, August 19–25, 2001). Edited by Petr Simon. Published April 2002 by Topology Atlas (electronic publication). Pp. 331-346; arXiv: math.GN/0204144

[13] A.M. Vershik, *The universal Urysohn space, Gromov's metric triples, and random metrics on the series of natural numbers*, Uspekhi Matem. Nauk **53** (1998), no. (5), pp. 57-64. English translation: Russian Math. Surveys **53** (1998), no. (5), pp. 921-928; Correction: Uspekhi Mat. Nauk **56** (2001), no. 5, p. 207; translation in Russian Math. Surveys **56** (2001), no. 5, p. 1015.

[14] A.M. Vershik, *Random metric spaces and the universal Urysohn space*, Fundamental Mathematics Today. 10th anniversary of the Independent Moscow University. MCCME Publ., 2002.

[15] A.M. Vershik, *Random metric space is universal Urysohn space*, Doklady RAN, 2002, **386**, no. 6, to appear.

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